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TWO-POINTS PROBLEM FOR AN EVOLUTION FIRST ORDER EQUATION IN BANACH SPACE

The two-point problem nonlocal problem for the first order differential evolution equation with an operator coefficient in a Banach space X is considered. An exponentially convergent algorithm is proposed and justified in assumption that the operator coefficient is strongly positive and some existence and uniqueness conditions are fulfilled. The algorithm provides exponentially convergence in time that in combination with fast algorithm on spatial variables can be efficient treating such problems.

1. Introduction

We consider two-point nonlocal problem for the first order differential evolutional equation with an operator coefficient in a Banach space X

$$\begin{aligned} &\frac{du(t)}{dt}+A_1(t)u(t)=f_1(t),\\ &u(-1)+\alpha u(1)=\varphi, \end{aligned} \tag{1.1}$$

where $A_1(t)$ is a densely defined closed (unbounded) operator with the domain $D\left(A_1\right)$ independent of t in the Banach space X, φ is given vector and $f_1(t)$ is given vector-valued function, $\alpha \in \mathbb{R}$. We suppose that the operator $A_1(t)$ is strongly positive and the following assumptions are fulfilled: $\exists c_k, \omega > 0$ such that

$$\left\|A_1^k(t)e^{-sA_1(t)}\right\|\le c_k s^{-k},\ s>0,\ k>0 \tag{1.2}$$

$$\left\|e^{-sA_1(t)}\right\|\le e^{-\omega s}\ \ \forall\ s,t\in[-1,1] \tag{1.3}$$

$$\left\|\left(A_1(t)-A_1(s)\right)A_1^{-\gamma}(t)\right\|\le L_{1,\gamma}\left|t-s\right|,\ \forall t,s,0\le\lambda<1, \tag{1.4}$$

$$\left\|A_1^{-\gamma}(t)A_1^{-\gamma}(s)-I\right\|\le L_\gamma\left|t-s\right|,\ \forall t,s\in\left[-1,1\right]. \tag{1.5}$$

$$f_1(t)\in C\left(\left[-1,1\right];X\right) \tag{1.6}$$

2. Numerical algorithm

We use the approach developed in [1] and [2] to construct numerical method for solving problem (1.1). We choose a mesh $\omega_k=\left\{t_k,\ k=0,...,n\right\}$ of $n+1$ various points on $[-1,1]$ that are Chebyshev-Gauss-Lobatto nodes $t_k=\cos\left(\frac{n-k}{n}\pi\right)$, and set $\tau_k=t_k-t_{k-1}$. Let

$$\bar{A}(t) = A_k = A(t_k), t \in \left(t_{k-1}, t_k\right], k = 1,\dots,n,$$
$$A_0 = A(-1) \tag{2.2}$$

We rewrite the problem (2.1) in the equation form

$$\frac{dv(t)}{dt} + \bar{A}(t)\,v(t) = \left(\bar{A}(t) - A(t)\right)v(t) + f(t), t \in (-1,1)$$
$$u(-1) + \alpha u(1) = \varphi, \tag{2.3}$$

On each subinterval we can write down the equivalent to (2.3) integral equation

$$v(t) = e^{-A_k(t-t_{k-1})}v(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{-A_k(t-s)}\left(A_1 - A(t)\right)v(s)ds + \int_{t_{k-1}}^{t_k} e^{-A_k(t-s)}f(s)ds,$$
$$t \in \left[t_{k-1}, t_k\right], k = 2,\dots,n, \tag{2.4}$$

$$v(t) = e^{-A_1(t+1)}(\varphi - \alpha v(1)) + \int_{-1}^{t} e^{-A_1(t-s)}\left(A_1 - A(t)\right)v(s)ds + \int_{-1}^{t} e^{-A_1(t-s)}f(s)ds,$$
$$t \in \left[-1, t_1\right] \tag{2.5}$$

Let

$$P_n(t;v) = P_n v = \sum_{j=0}^{n} v(t_j)L_{j,n}(t) \tag{2.6}$$

be the interpolation polynomial for $v(t)$ on the mash $\omega_n, x = \left(x_0,\dots,x_n\right), x_i \in X$ given vector and

$$P_n(t;y) = P_n x = \sum_{j=0}^{n} x_j L_{j,n}(t) \tag{2.7}$$

The polynomial that interpolate x, where

$$L_{j,n}(s) = \frac{T_n'(s)(1-s^2)}{\frac{d}{ds}\left(\left(1-s^2\right)T_n'(s)\right)\Big|_{s=s_j}\left(s - s_j\right)},\ j = 0,\dots,n$$

are the Lagrange fundamental polynomials. Substituting $P_n(s;x)$ for $v(s), x_k$ for $v(t_k)$ and then setting $t = t_k$ in (2.4) we arrive at the following system of linear equations with respect to the unknown x_k

$$x_0 + \alpha x_n = \varphi,$$
$$x_k = e^{-A_k \tau_k} x_k + \sum_{j=0}^{n} \alpha_{k_j} x_j + \phi_k, \; k = 1,\dots,n, \tag{2.8}$$

wich represents our algorithm. Here we use the notation

$$\alpha_{kj} = \int_{t_{k-1}}^{t_k} e^{-A_k (t_k - s)} \left(A_k - A(s) \right) L_{j,n}(s) ds + \int_{t_{k-1}}^{t_k} e^{-A_k (t-s)} f(s) ds,$$
$$\varphi_k = \int_{t_{k-1}}^{t_k} e^{-A_k (t-s)} f(s) ds, \; k = 1,\dots,n, \; j = 0,\dots,n, \tag{2.9}$$

and suppose that we have algorithm to compute these coefficients.
For the error $z = \left(z_1, \dots, z_n \right)$, with $z_k = v(t_k) - x_k$ we have the relations

$$z_0 + \alpha z_n = 0,$$
$$z_k = e^{-A_k \tau_k} z_{k-1} + \sum_{j=0}^{n} \alpha_{k_j} z_j + \psi_k, \; k = 1,\dots,n, \tag{2.10}$$

Where

$$\psi_k = \int_{t_{k-1}}^{t_k} e^{-A_k (t_k - s)} \left(A_k - A(s) \right) \left(v(s) - P_n(s; v) \right) ds, \; k = 1,\dots,n \tag{2.11}$$

In order to represent algorithm (2.8) in a block-matrix form we introduce the matrix

$$S = \begin{pmatrix} I & 0 & 0 & . & . & . & 0 & \alpha\sigma_0 \\ -\sigma_1 & I & 0 & . & . & . & 0 & 0 \\ 0 & -\sigma_2 & I & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -\sigma_n & I \end{pmatrix} \tag{2.12}$$

where $\sigma_0 = A_0^{\gamma} A_n^{-\gamma}, \sigma_k = e^{A_k \tau_k} A_k^{\gamma} A_{k-1}^{-\gamma}, k = 1,\dots,n$, the matrix $B = \left\{ \bar{\alpha}_{k,j} \right\}_{k,j=0}^{n}$ with

$\bar{\alpha}_{k,j} = A_k^{\gamma} \alpha_{k,j} A_j^{-\gamma}, k = 1,\dots,n; j = 0,\dots,n$ and $\alpha_{0,j} = 0, j = 0,\dots,n$, the vectors

$$\bar{x} = \begin{pmatrix} A_0^{\gamma} x_0 \\ A_1^{\gamma} x_1 \\ . \\ . \\ . \\ A_n^{\gamma} x_n \end{pmatrix}, \phi = \begin{pmatrix} A_0^{\gamma} \varphi \\ A_1^{\gamma} \phi_1 \\ . \\ . \\ . \\ A_n^{\gamma} \phi_n \end{pmatrix}, \bar{z} = \begin{pmatrix} A_0^{\gamma} z_0 \\ A_1^{\gamma} z_1 \\ . \\ . \\ . \\ A_n^{\gamma} z_n \end{pmatrix}, \psi = \begin{pmatrix} 0 \\ A_1^{\gamma} \psi_1 \\ . \\ . \\ . \\ A_n^{\gamma} \psi_n \end{pmatrix} \tag{2.13}$$

We multiply the equations in (2.8) and the equation in (2.10) by $A_k^{\lambda}, k=0,...,n$ and obtain

$$
\begin{aligned}
&A_0^{\gamma} x_0 + \alpha A_0^{\gamma} x_n = A_0^{\gamma} \varphi, \\
&A_k^{\gamma} x_k = e^{-A_k \tau_k} A_k^{\gamma} x_{k-1} + \sum_{j=0}^{n} \bar{\alpha}_{k_j} A_j^{\gamma} x_j + A_k^{\gamma} \phi_k,\ k=1,...,n,
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&A_0^{\gamma} z_0 + \alpha A_0^{\gamma} z_n = 0, \\
&A_k^{\gamma} z_k = e^{-A_k \tau_k} A_k^{\gamma} z_{k-1} + \sum_{j=0}^{n} \bar{\alpha}_{k_j} A_j^{\gamma} z_j + A_k^{\gamma} \psi_k,\ k=1,...,n,
\end{aligned}
\tag{2.16}
$$

We have constructed an exponentially convergent algorithm. This algorithm leads to a system of linear equations that can be solved by fixed-point iteration. The algorithm provides exponentially convergence in time that in combination with fast algorithms on spatial variables variables can be efficient treating such problems. The following theorem is valid.

Theorem 1. Under the given assumptions there exists a positive constant c such that for the error of our method the following estimate

$$
\|\tilde{z}\| \le c n^{\gamma-1} \ln(n) E_n\left(A_0^{\gamma} v\right) \tag{2.17}
$$

holds true for n large enough, where v is the solution of (1), with the error of the best approximation of u by polynomials of degree not greater then n

$$
\left\|E_n\left(u\right)\right\| = \inf_{p\in\Pi_n} \max_{t\in[-1,1]} \|u(t)-p(t)\|. \tag{2.18}
$$

The efficiency of the proposed algorithm is demonstrated by numerical examples.

References

[1] *I.P. Gavrilyuk and V.L. Makarov.* Algorithms without accuracy saturation for evolution equations in Hilbert and Banach spaces. Math. Comp., 74:555{583, 2005.

[2] *T.Ju.Bohonova, I.P.Gavrilyuk, V.L.Makarov, and V.B.Vasylyk.* Exponentially convergent Duhamel like algorithms for differential equations with an operator coefficient possessing a variable domain in a Banach space. SIAM J. Numer. Anal., 46(5):365{396, 2007/08.